Scaling laws, from Perceptrons to Deep networks

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Outline of the talk

1. Review on neural scaling law
   - Empirical findings on neural scaling laws
   - Two models to predict power-laws exponents
   - Discussion (1^o part)

2. Our results (with Dario Bocchi and Matteo Negri)
   - Simple perceptron model
   - Experiments on deep networks
   - Discussion (2^o part)

Part IA: Empirical findings

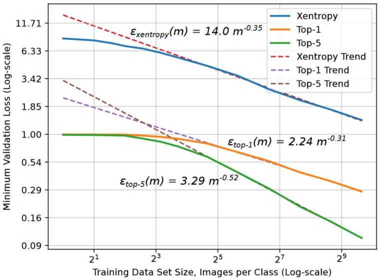
- Neural scaling laws phenomenology
- Why they motivated large scale LLMs like GPT-3/4
- How to use them to optimize compute cost

Glossary

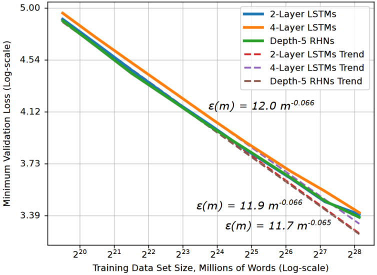
- P: number of training data
- N: total number of learnable parameters
- $\mathscr{L}$: generalization loss, i.e. cross-entropy in classification
- ε: generalization error

Hestness et al (2017):
Deep Learning Scaling is Predictable, Empirically

ResNet, image classification



LLM, next word prediction



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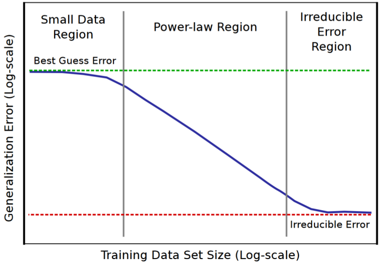

Power law in intermediate regime:

$$\mathscr{L}(P) \sim cP^{-\gamma}$$

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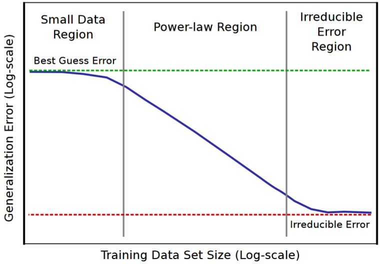

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Empirical properties of curves for model tested:

- Power laws in all domains tested
- Exponent γ depends on task/dataset
- Architectures change mainly constant c
- Same for optimizers (SGD, Adam ..)

Rosenfeld et al. (2020): A Constructive Prediction of the Generalization Error Across Scales

Two different scaling laws:

$$\varepsilon(N,P) \approx \begin{cases} aP^{-\alpha} + c_P(N) & \text{(data scaling at fixed model)} \\ bN^{-\beta} + c_N(P) & \text{(model scaling at fixed dataset)} \end{cases}$$

($P =$ #data, $N =$ #parameters)

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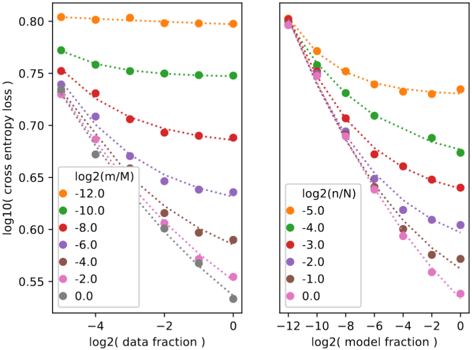

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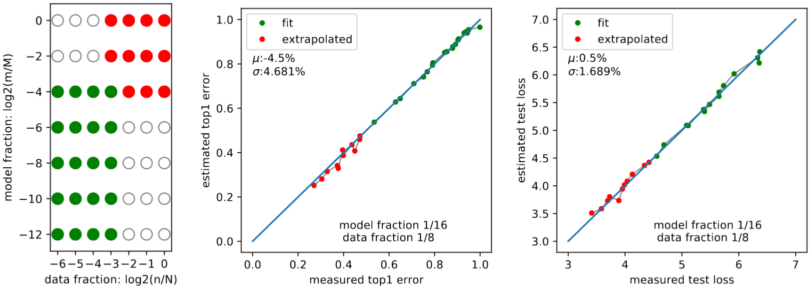

(a) Illustration. (b) Extrapolation on ImageNet (c) Extrapolation on WikiText-103.

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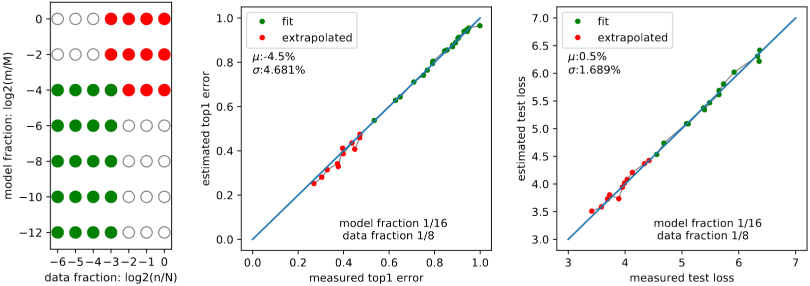
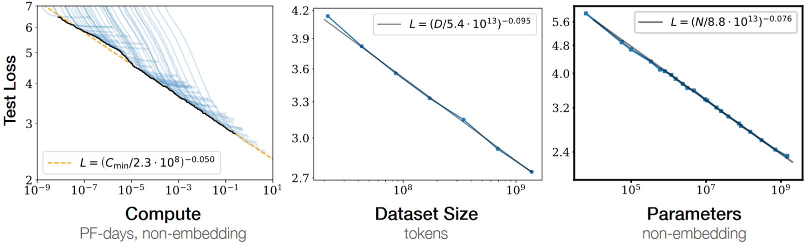


(a) Illustration. (b) Extrapolation on ImageNet (c) Extrapolation on WikiText-103.

$\Rightarrow$ small P,N models capable of predicting large P,N models

Kaplan et al (2020):
Scaling laws for neural language models

Almost perfect scaling laws in GPT models across many magnitudes



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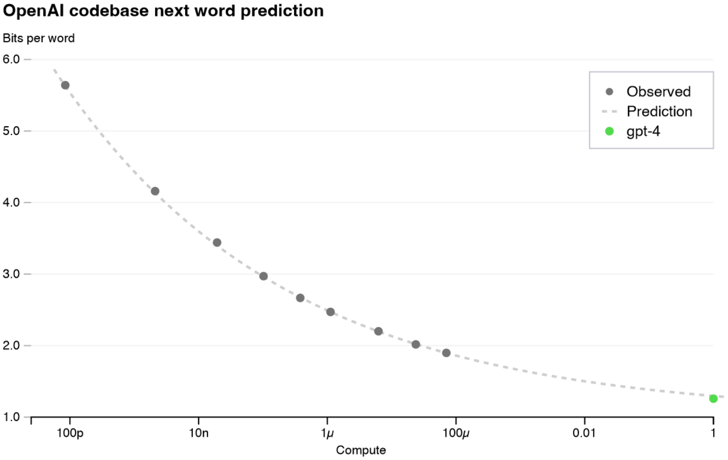
"Scaling is all you need"

Kaplan et al (2020): Scaling laws for neural language models

All those results motivated extreme P, N scaling $\Rightarrow$ GPT-3/4 models



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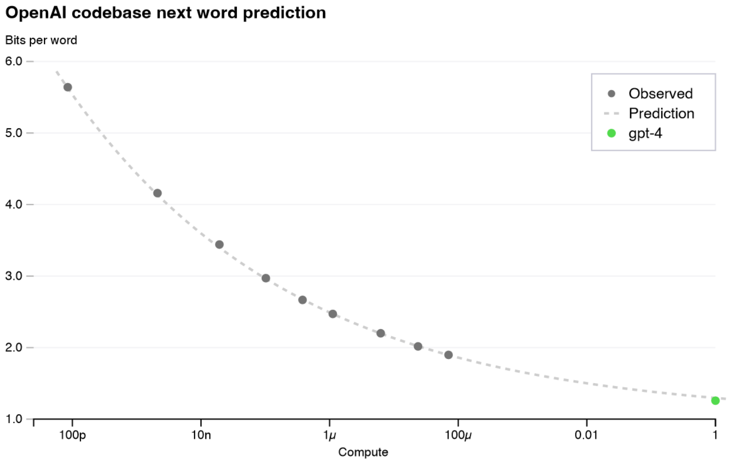

Smaller models fit predicted GPT-4 loss

Hoffmann et al. (2022): Training Compute-Optimal Large Language Models

Given an available compute C, what is best choice of N, P?

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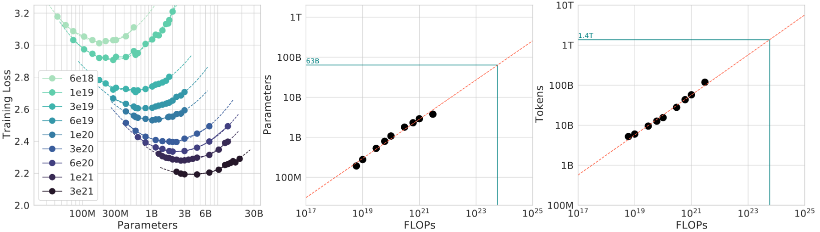

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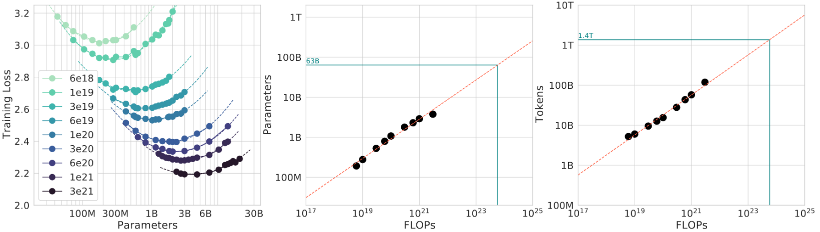

$$\Rightarrow P_{\text{opt}}(C), N_{\text{opt}}(C) \text{ both } \sim C^{0.5}$$

Chinchilla scaling law

Summary of empirical results

1. Loss/error scales as $\varepsilon(N,P) = aP^{-\alpha} + bN^{-\beta} + c_\infty$
2. Exponents robusts wrt most of details of training and architectures
3. Exponents found $\in [0.05, 0.5]$
4. Best strategy given a compute C to scale $P, N \sim C^{0.5}$

Part IB. Two attempts to explain exponents: geometric bounds and DMFT models

Bahri et al. (2021): Explaining Neural Scaling Laws

Idea:

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Case $\mathscr{L}(P) - \mathscr{L}(\infty) = \Delta(P)$:

1. **Underparametrized** ($P \gg N \gg 1$): **variance** dominates $\Delta(P) \sim c_{\text{var}} P^{-1}$ (infinite limit + corrections)

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Case $\mathscr{L}(N) - \mathscr{L}(\infty) = \Delta(N)$:

1. **Overparametrized** ($N \gg P \gg 1$): **variance** dominates
   $\Delta(N) \sim c_{\text{var}} N^{-1}$ ($N^{-1/2}$ deep case) (infinite limit + corrections)
2. **Underparametrized** ($P \gg N \gg 1$): **bias** dominates
   $\Delta(N) \sim c_{\text{bias}} N^{-\alpha_{\text{bias}}}$

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1. Exponents $\{-1, -1/2\}$ in variance-dominated regimes
2. Different exponents in bias-dominated regimes

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$$\boxed{\Delta(P) \sim P^{-1/d} \;\; ; \;\; \Delta(N) \sim N^{-1/d}}$$

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Assuming:

- Data lie on d-dimensional hidden manifold
- Teacher-student: $y = F(x)$ and $\hat{y} = f(x)$

Bahri et al. (2021): Explaining Neural Scaling Laws

Analytical model: linear random features

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- Teacher

$$F(x) = \sum_{M=1}^{S} \omega_M F_M(x)$$

Bahri et al. (2021): Explaining Neural Scaling Laws

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- Student features $f_\mu \in P$-dimensional subspace of teacher features

Analytical model: linear random features

Key ingredient: power-laws in features and data

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- Feature-feature second moment matrix:

$$\mathscr{C} = \mathbb{E}_x[F(x)F^T(x)]$$

- Data-data second moment matrix:

$$\mathscr{K}(x,x') = \frac{1}{S}F^T(x)F(x')$$

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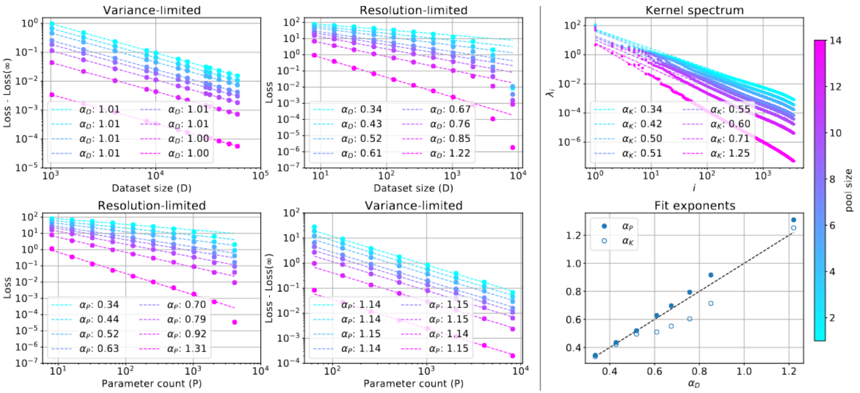
2. $\alpha_K \sim 1/d$

Bahri et al. (2021): Explaining Neural Scaling Laws

Result: linear random features



Bordelon et al. (2024):
A Dynamical Model of Neural Scaling Laws

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- All consistent at $t \to \infty$ with previous results

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- Data $\mathbf{x} \in \mathbb{R}^N$ drawn $\mathbf{x} \sim p(\mathbf{x})$
- Teacher from fixed features $\psi(\mathbf{x}) \in \mathbb{R}^M$ + noise:

$$y(\mathbf{x}) = \frac{1}{\sqrt{M}} \mathbf{w}^* \cdot \psi(\mathbf{x}) + \sigma \varepsilon(\mathbf{x})$$

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- Student is a lower-dimensional projection of features $\mathbf{A}\psi(\mathbf{x})$ where $\mathbf{A} \in \mathbb{R}^{N \times M}$, A_{ij} i.i.d.

$$f(\mathbf{x}) = \frac{1}{\sqrt{N}} \mathbf{w} \cdot \mathbf{A}\psi(\mathbf{x})$$

Bordelon et al. (2024): A Dynamical Model of Neural Scaling Laws

Assumption: power-law features + data

1. Given $\langle \psi_k(\mathbf{x}) \psi_l(\mathbf{x}) \rangle_{\mathbf{x} \sim p(\mathbf{x})} = \delta_{kl} \lambda_k$ (fixed)

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   $\Rightarrow$ assume $(\omega_k^*)^2 \lambda_k \sim k^{-a}$

   - $(\omega_k^*)^2 \lambda_k$ controls generalization error per mode
   - Large a $\Rightarrow$ target error concentrated in first modes $\Rightarrow$ easy task

Bordelon et al. (2024):
A Dynamical Model of Neural Scaling Laws

DMFT results

(1) *Bottleneck scalings*

$$\mathscr{L}(t,P,N) \approx \begin{cases} t^{-\frac{a-1}{b}}, & P,N \to \infty \quad \text{(Time)}, \\ P^{-\min\{a-1,2b\}}, & t,N \to \infty \quad \text{(Data)}, \\ N^{-\min\{a-1,2b\}}, & t,P \to \infty \quad \text{(Model)}. \end{cases}$$

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(2) Compute optimal

- Compute optimal time-size: $\boxed{t \sim C^{\frac{b}{1+b}}\ ,\ N \sim C^{\frac{1}{1+b}}}$

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   Recent attempts with feature learning:

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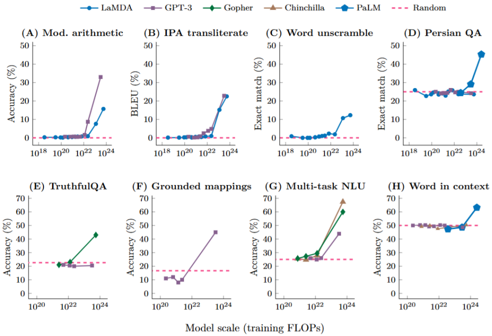
2. Different (complicated) tasks produce "phase-transitions"

   Wei et al., (2022): Emergent Abilities of Large Language Models



References

1. Hestness et al (2017): Deep Learning Scaling is Predictable, Empirically
2. Rosenfeld et al. (2020): A Constructive Prediction of the Generalization Error Across Scales
3. Kaplan et al (2020): Scaling laws for neural language models
4. Bahri et al. (2021): Explaining Neural Scaling Laws
5. Hoffmann et al. (2022): Training Compute-Optimal Large Language Models
6. Maloney et al. (2022): A Solvable Model of Neural Scaling Laws
7. Wei et al., (2022): Emergent Abilities of Large Language Models
8. Bordelon et al. (2024): A Dynamical Model of Neural Scaling Laws
9. Bordelon et al. (2025) How Feature Learning Can Improve Neural Scaling Laws
10. Defilippis et al. (2025) Scaling Laws and Spectra of Shallow Neural Networks in the Feature Learning Regime

Part II: Our work

Implicit bias produces neural scaling laws in learning curves, from perceptrons to deep networks

Francesco D'Amico[1,2]*, **Dario Bocchi**[1,2]*, **Matteo Negri**[1,2]

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Outline:

1. We show two new scalings laws in a simple Perceptron model

Part II: Our work

Implicit bias produces neural scaling laws in learning curves, from perceptrons to deep networks

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1. We show two new scalings laws in a simple Perceptron model
2. These new laws combined reproduce $\varepsilon \sim P^{-\gamma}$ scaling law
3. Valid empirically for Deep Nets in real image classification

Perceptron model

- *Student* perceptron $\boldsymbol{w} \in \mathbb{R}^N$, *Teacher* perceptron $\boldsymbol{w}^* \in \mathbb{R}^N$

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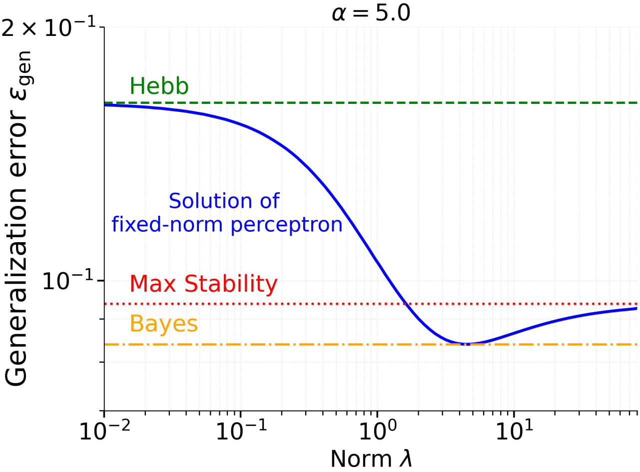
- Spherical weights $\|\boldsymbol{w}^*\|^2 = \|\boldsymbol{w}\|^2 = \lambda N$
- Cross-entropy (Pseudo-likelihood) Loss:

$$L(\boldsymbol{w};\lambda) = -\left[\sum_{\mu=1}^{P} \Delta^\mu - \log 2\cosh\left(\Delta^\mu\right)\right] = \sum_{\mu=1}^{P} V(\Delta^\mu)$$

where *margins*

$$\Delta^\mu \equiv y^\mu \left(\frac{\boldsymbol{w}\cdot\boldsymbol{x}^\mu}{\sqrt{\lambda N}}\right)$$

Solution at fixed α interpolates known learning rules



Unbounded norm perceptrons $\approx$ fixed-norm

- Norm $\lambda(t)$ increases monotonically for GD, Soudry et al., (2018)

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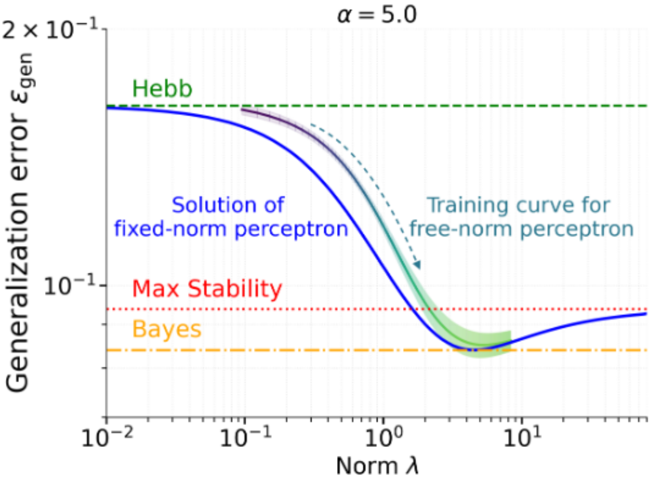

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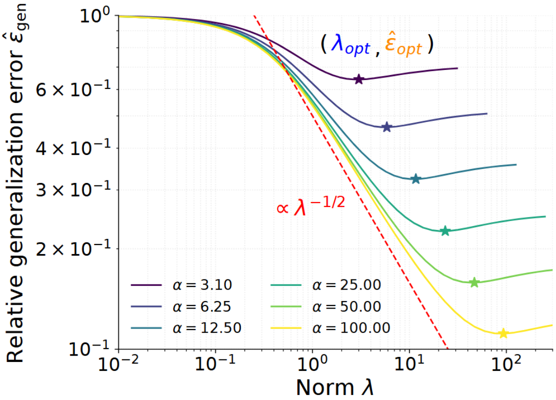

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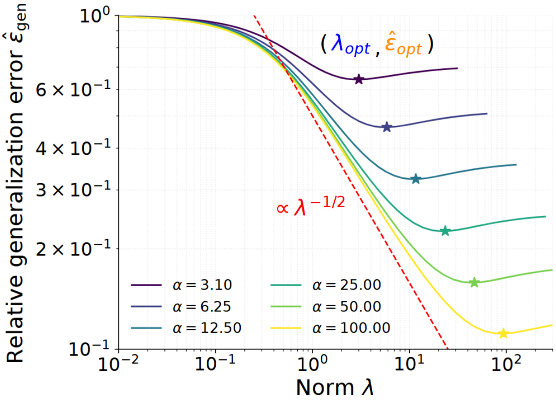

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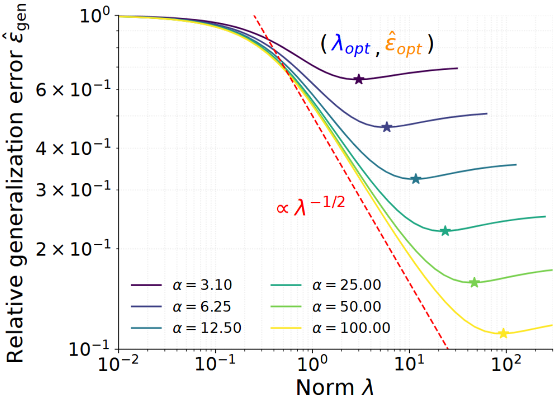

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Result (2): collapse on a master curve Φ

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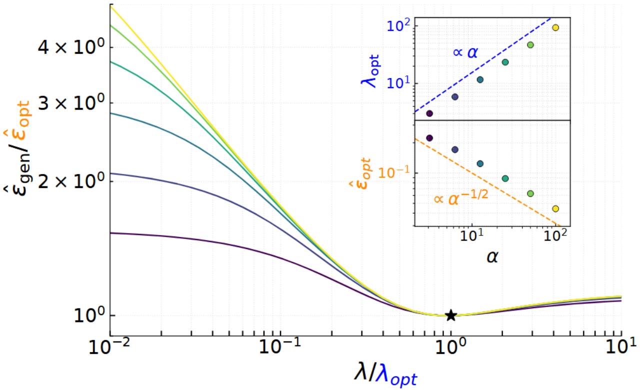

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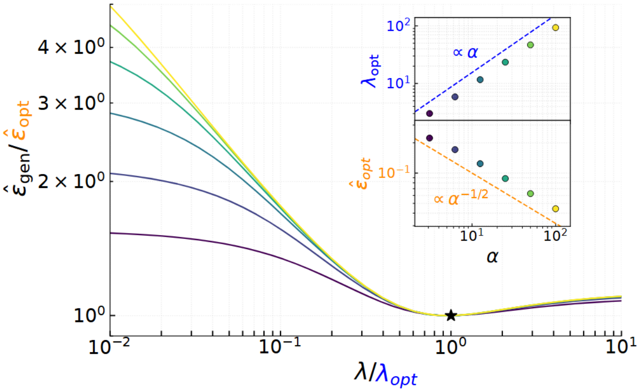

Curves converge to a master curve for $\alpha \gg 1$: $\Phi_\alpha \to \Phi$

Result (3): predict neural scaling law

1. $\hat{\varepsilon}_{\text{gen}} \sim k_1 \lambda^{-\gamma_1}$ for $\lambda < \lambda_{elbow}(\alpha)$
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$\hat{\varepsilon}_{\text{gen}} \sim \alpha^{-\gamma}$, with $\boxed{\gamma = \gamma_1 \gamma_2}$

Does the theory also apply to deep networks?

Architectures:

- Convolutional Neural Networks (CNN)
- Residual Neural Networks (ResNet)
- Vision Transformers (ViT)

Datasets:

- MNIST (greyscale digits, 10 classes)
- CIFAR10 (RGB images, 10 classes)
- CIFAR100 (RGB images, 100 classes)

Norm in deep networks: Bartlett et al. (2017) Spectrally-normalized margin bounds for neural networks

Spectral Complexity norm for a L-layer deep net with matrices A_i:

- ρ_i Lipschitz constant of layer i activation function
- $\|\cdot\|_\sigma$ biggest singular value (spectral norm)
- $\|\cdot\|_{2,1}$ sum of ℓ_2 norms of columns
- M_i reference matrix (can be $= \mathbf{0}$)

$$R_A = \left(\prod_{i=1}^{L} \rho_i \|A_i\|_\sigma \right) \left(\sum_{i=1}^{L} \frac{\|A_i^\top - M_i^\top\|_{2,1}^{2/3}}{\|A_i\|_\sigma^{2/3}} \right)^{3/2}$$

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Maximum expansion **Effective rank**

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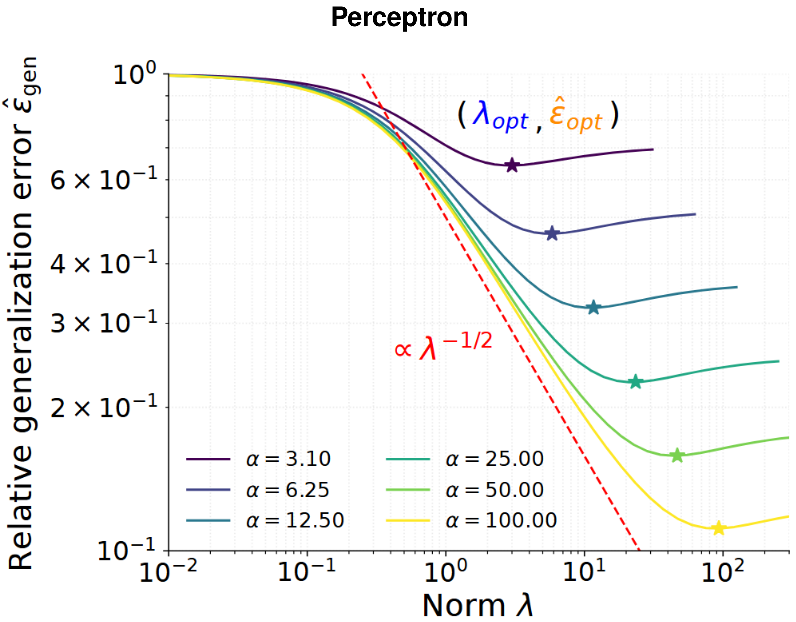

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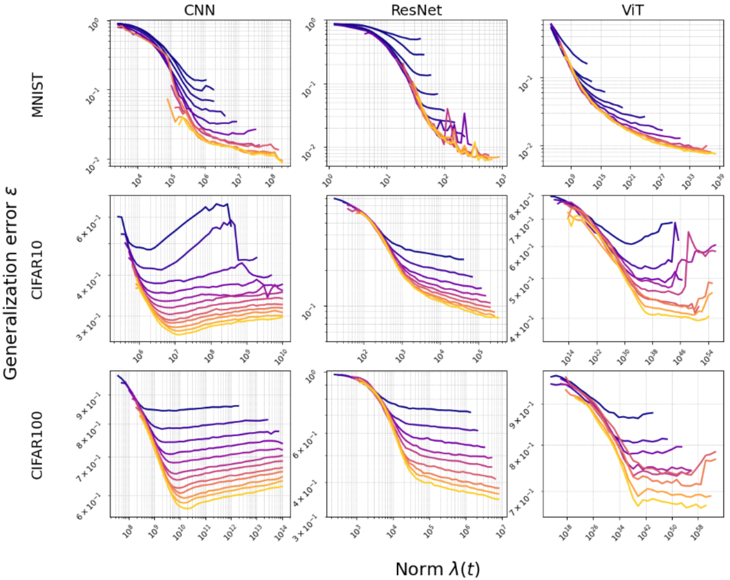
Deep Networks



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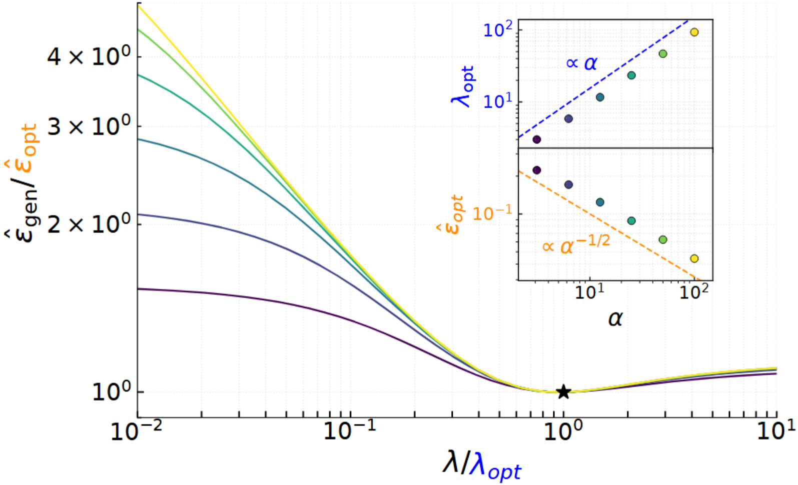

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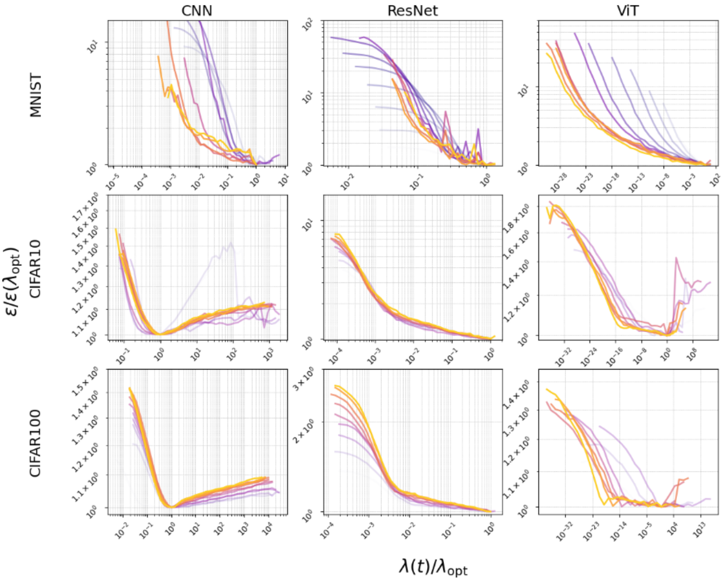

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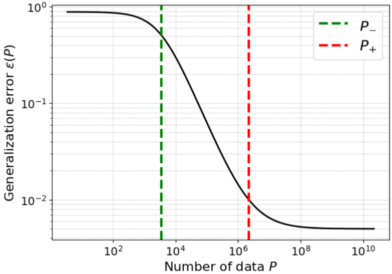

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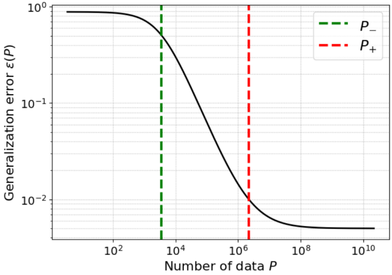

Hestness et al (2017) empirical curve

Model	Dataset	γ_{pred}	γ_{meas}	σ
CNN	MNIST	0.60	0.55	0.09
CNN	CIFAR10	0.28	0.25	0.07
CNN	CIFAR100	0.16	0.16	0.03
ResNet	MNIST	0.57	0.69	0.08
ResNet	CIFAR10	0.54	0.56	0.04
ResNet	CIFAR100	0.31	0.37	0.03
ViT	MNIST	0.47	0.54	0.03
ViT	CIFAR10	0.23	0.21	0.03
ViT	CIFAR100	0.14	0.12	0.04

$\gamma_1 \gamma_2$ **compatible with** γ_{meas}

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Numerically we tested

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3. In (3) $\gamma_1\gamma_2 \neq \gamma \Rightarrow$ Spectral complexity is "special"

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- Only image classification

  Extension: LLMs (i.e. Maloney et al. (2022) A Solvable Model of Neural Scaling Laws)

Thank you for attention!

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October 23, 2025

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Bahri et al. (2021): Explaining Neural Scaling Laws

Idea:

Why $-1/d$ exponents? Arguments for *bounds*

1. Scaling in P (overparametrized):
   Distance of test points to closest training point $\mathcal{O}(P^{-1/d})$
2. Scaling in N (underparametrized):
   1. Take N *anchor* points $I = \{\mathbf{x}\}_{1,\ldots,N}$ from the huge dataset.
   2. f approximates F piecewise with N regions, centered on I points.
   3. Distance of test points to closest I: $\mathcal{O}(N^{-1/d})$